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Analytic results for the scaling behaviour of a piecewise-linear map of the circle

Jacob Wilbrink†

Twente University of Technology, Center for Theoretical Physics, PO Box 217, 7500 AE Enschede, The Netherlands

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Abstract. For the piecewise-linear circle map $\theta \to \theta'$, with $\theta' \equiv \theta + \Omega - K(\frac{1}{2} - |\theta(\text{mod } 1) - \frac{1}{2}|)$ the parameter values Ω_i at which a periodic orbit, starting at $\theta = 0$ with winding number F_i/F_{i+1}, where F_i is the ith Fibonacci number, exists, are calculated analytically. These calculations are done at two K values, $K = 1$, the critical case, and at $K = \kappa < 1$ (where $\ln(1-\kappa)/\ln(1+\kappa) = -(1+\sqrt{5})/2$). At $K = \kappa$ the usual scaling behaviour for a smooth subcritical map is found, i.e. the same δ as Shenker found numerically. However, at $K = 1$ a different critical δ value than is usually found numerically for smooth maps is calculated analytically for this piecewise-linear map.

1. Introduction

Recently much work has been done on smooth circle maps. It was mainly stimulated by the fact that in circle maps the transition from 'rotation-like' behaviour to 'chaotic' behaviour is an analogue (Feigenbaum *et al* 1982) of a particular route to chaos: quasi-periodic behaviour followed by broadband noise. This scenario is often observed in experiments (Swinney and Gollub 1978).

I study the piecewise-linear circle map $\theta' = T_0\theta$, with

$$T\colon\ \theta' = \theta + \Omega - KD(\theta) \tag{1}$$

$$D(\theta) \equiv \begin{cases} \bar\theta & 0 \leqslant \bar\theta < (a-1)/a \\ (a-1)(1-\bar\theta) & (a-1)/a \leqslant \bar\theta < 1 \end{cases} \tag{2}$$

$$a > 1 \qquad \bar\theta \equiv \theta \text{ modulo } 1 \qquad 0 \leqslant \bar\theta < 1$$

as plotted in figure 1. The cusp is located at $\bar\theta = (a-1)/a$. Note that $T_0(\theta+1) = T_0\theta + 1$ and $T_0\theta|_{\text{mod } 1}$ is a map on the circle. An orbit is a sequence of subsequent θ values, θ, $T_0\theta$, $T^2{}_0\theta, \ldots$, generated by the mapping.

This paper is confined to orbits starting at $\theta = 0$. The orbit has (rational) winding number $\rho = F/G$ if

$$T^G{}_0 0 = F. \tag{3}$$

Actually I wish to find orbits with irrational winding number ρ, equal to the golden mean $W_\infty \equiv (\sqrt{5}-1)/2$, cf Shenker (1982). This is done by approximations with the

† Present address: Physics Board, University of California, Santa Cruz, CA 95064, USA.

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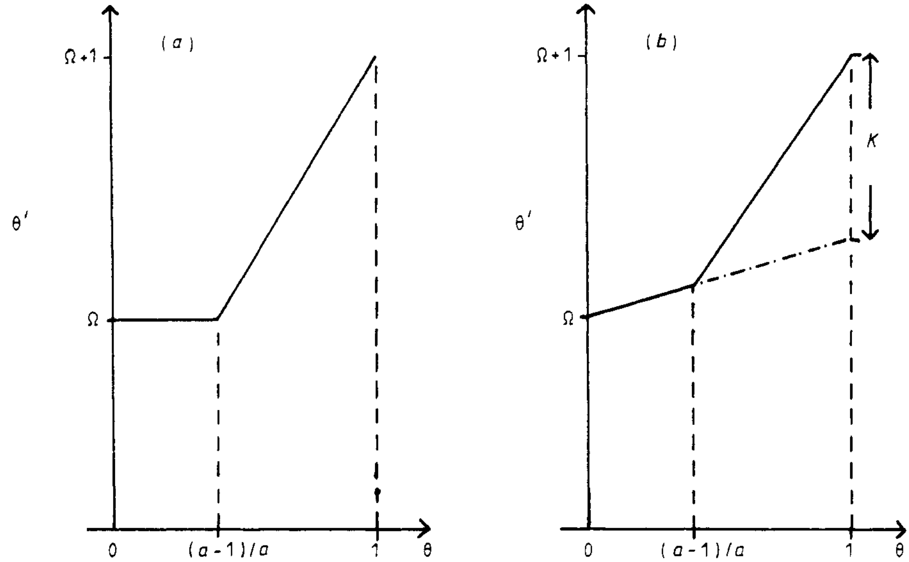

Figure 1. The piecewise-linear mapping T, from (1), plotted as a function of θ, at (a) $K = 1.0$ and (b) $K < 1.0$. Ω is a parameter in equation (1).

rationals

$$W_i \equiv F_i / F_{i+1} \tag{4}$$

where F_i is the ith Fibonacci number ($F_0 \equiv 0$, $F_1 = 1$, $F_{i+1} \equiv F_i + F_{i-1}$). These W_i yield the best rational approximants to W_∞ (Shenker 1982, Niven 1956). The main technical problem is to calculate the parameter values $\Omega_i(K)$, for (1), at which orbits with $\rho = W_i$ exist.

A derived quantity of importance is the rate δ at which the $\Omega_i(K)$ converge, and its approximation

$$\delta_i(K) \equiv [\Omega_{i-1}(K) - \Omega_i(K)]/[\Omega_i(K) - \Omega_{i+1}(K)]. \tag{5}$$

Shenker (1982) numerically calculated δ_i and Ω_i for two smooth circle maps. I obtain these quantities analytically for the piecewise-linear map (1) at two K values.

The circle maps studied here and in Shenker (1982) have a critical K value, $K = 1$, at which there is a transition to chaotic behaviour, similar to the transition to broadband noise in the scenario described above. In the subcritical case ($K < 1$) these analytical results agree with Shenker's (1982) numerical value, $\delta = -2.6180\ldots$.

Ostlund *et al* (1983) point out that

$$\delta = -W_\infty^{-2}$$

for an analytic diffeomorphism; this also holds for this non-analytic map. In the critical case my analytical result is different from Shenker's result, apparently due to the existence of a finite region of slope 0 (see figure 1(a)).

In § 2 the method for recursively obtaining the $\Omega_i(K)$ values is described. In §§ 3 and 4 the Ω_i and δ_i are calculated for $K = 1$ and $K = \kappa$, where κ is the solution of

$$\ln(1-\kappa)/\ln(1+\kappa) = -(1+\sqrt{5})/2. \tag{6}$$

2. A recursive method to calculate $\Omega_i(K)$

The problem is to calculate the values Ω_k such that

$$T_k{}^{F_{k+1}}{}_0 0 = F_k \tag{7}$$

where T_k denotes the map T (as defined in (1)) with the parameter value $\Omega = \Omega_k$.

I shall obtain a recursion relation for those Ω_k,

$$\Omega_k = \Omega_k(\Omega_{k-1}, \Omega_{k-2}). \tag{8}$$

Use will be made of the fact that $T_i{}^j{}_0 0$ and $T_{i-1}{}^j{}_0 0$ are both in the same unit interval, e.g. $[n, n+1]$, $n \in N$, and on the same side of the cusp in $D(\theta)$, for all j with $j < F_i$. A proof follows later in this section.

From this result it follows that $T_i{}^j{}_0 0$ and $T_{i-1}{}^j{}_0 0$ are virtually the same; when $j \leqslant F_i$ they merely differ in their Ω values. Hence

$$T_i{}^j{}_0 0 - T_{i-1}{}^j{}_0 0 = B(j)(\Omega_i - \Omega_{i-1}) \tag{9}$$

where $B(j)$ is some proportionality constant to be calculated in §§ 3 and 4, which is not dependent on the Ω. Similarly $T_{i-1}{}^j{}_0 0$ and $T_{i-2}{}^j{}_0 0$ are in the same interval, for all j with $j < F_{i-1}$. Adding equation (9) to itself, at one lower i value, then yields

$$T_i{}^j{}_0 0 - T_{i-2}{}^j{}_0 0 = B(j)[\Omega_i - \Omega_{i-2}] \tag{10}$$

for $j \leqslant F_{i-1}$. Assuming equation (7) holds at $k = i-1$ and $i-2$, I derive a recursion relation for the Ω (8) such that (7) also holds at $k = i$. For $k = i$, equation (7) can be written as

$$T_i{}^{F_{i+1}}{}_0 0 = T_i{}^{F_{i-1}}{}_0 T_i{}^{F_i}{}_0 0 = F_i \tag{11}$$

since $F_{i+1} \equiv F_i + F_{i-1}$. Combining (9) and (7) at $k = i-1$ yields

$$T_i{}^{F_i}{}_0 0 = B(F_i)[\Omega_i - \Omega_{i-1}] + F_{i-1} \equiv \Phi_i + F_{i-1} \tag{12}$$

where I define a new quantity Φ_i.

As a result the second term of (11) becomes

$$T_i^{F_{i-1}}{}_0 T_i^{F_i}{}_0 0 = T_i^{F_{i-1}}{}_0(\Phi_i + F_{i-1}) = T_i^{F_{i-1}}{}_0 \Phi_i + F_{i-1} \tag{13}$$

due to the modulo counting in $D(\theta)$, cf (2). Later in this section I prove that Φ_i is so small that $T_i{}^j{}_0 \Phi_i$ will be in the same unit interval as $T_i{}^j{}_0 0$, and on the same side of the cusp in $D(\theta)$ (2), for all j with $j < F_{i-1}$. Hence

$$T_i{}^{F_{i-1}}{}_0 \Phi_i = T_i{}^{F_{i-1}}{}_0 0 + C(F_{i-1})\Phi_i \tag{14}$$

where $C(F_{i-1})$ is the product of the slopes in figure 1 each time T has been applied. Substitution of (14) and (10), at $j = F_{i-1}$, into (13) finally yields

$$T_i{}^{F_{i-1}}{}_0 T_i{}^{F_i}{}_0 0 = B(F_{i-1})(\Omega_i - \Omega_{i-2}) + C(F_{i-1})\Phi_i + F_i. \tag{15}$$

Comparison with (11) shows that

$$B(F_{i-1})(\Omega_i - \Omega_{i-2}) + C(F_{i-1})B(F_i)(\Omega_i - \Omega_{i-1}) = 0 \tag{16}$$

using the Φ_i definition (12).

Introducing some new notation,

$$B_{i-1} \equiv B(F_i) \qquad A_i \equiv \Omega_i B_i \tag{17}$$

equation (16) can be written as

$$[B_{i-2}+C(F_{i-1})B_{i-1}]\Omega_i = A_{i-2}+C(F_{i-1})A_{i-1}.$$

Hence I obtain for Ω_i

$$\Omega_i(=A_i/B_i) = [A_{i-2}+C(F_{i-1})A_{i-1}]/[B_{i-2}+C(F_{i-1})B_{i-1}]. \tag{18}$$

This equation still has two unknowns: A_i and B_i. From (9) a second equation can be derived which gives a recursion relation for B_i. Using (9) at various i and j values, and also using (7) and (14), it is straightforward to derive

$$B_i(\Omega_{i+1}-\Omega_i) = B_{i-2}(\Omega_{i+1}-\Omega_i)+C(F_{i-1})B_{i-1}(\Omega_{i+1}-\Omega_i) \tag{19}$$

and since

$$\Omega_{i+1}\neq\Omega_i$$

(19) can be written as

$$B_i = B_{i-2}+C(F_{i-1})B_{i-1} \tag{20a}$$

and, with (18), one finds

$$A_i = A_{i-2}+C(F_{i-1})A_{i-1}. \tag{20b}$$

In §§ 3 and 4 I calculate expressions for Ω_i (36) and (41) and δ_i (37) and (50) with the recursion relations (20).

Several technical proofs postponed from earlier in this section are provided now.

Result 1. $T_{i-1}{}^j{}_0 0$ and $T_i{}^j{}_0 0$ are in the same unit interval $[n, n+1]$, $n\in N$, with $j<F_{i+2}$.

Proof. It is proved by Kandanoff (1983) that the quantity

$$\Gamma_{P,Q}(\theta)\equiv[T_i{}^Q{}_0\theta-P-\theta]/[QW_i-P] \tag{21}$$

is greater than zero for all P, $Q(\in N)$ and θ for which the denominator does not vanish, i.e. $P/Q\neq W_i$.

The main idea of this proof is that under these conditions the numerator cannot vanish either since all orbits of T_i have winding number W_i. The numerator would vanish only if $P/Q=F_i/F_{i+1}$, which was excluded. In addition, note that $\Gamma_{P,Q}(\theta)$ is periodic in θ and continuous. Hence it will be positive for all θ when it is positive for one θ. Also it is easily seen from (21) that $\Gamma_{NP,NQ}(\theta)\to 1$ if $N\to\infty$, due to $\lim_{N\to\infty}[(T_i{}^{NQ}{}_0\theta)/(NQW_i+\theta)]=1$. So $\Gamma_{P,Q}(\theta)$ will always be positive (Kadanoff 1983). Hence $T_i{}^j{}_0 0$ is in the same interval as jF_i/F_{i+1}.

To show that jF_i/F_{i+1} and jF_{i-1}/F_i are in the same unit interval for $j<F_{i+2}$ I will look for the first time that this is not the case. (Here I assume i is even, proof for i odd is analogous.) I need the smallest integers n, j such that

$$jF_i/F_{i+1}<n<jF_{i-1}/F_i. \tag{22}$$

This can be rewritten, using $F_{i-1}F_{i+1}=F_i^2+1$, as

$$0<nF_{i+1}-jF_i<j/F_i. \tag{23}$$

The second term is an integer. When it takes the value 1, it can be rewritten, using $F_{i-1}F_{i+1}=F_i^2+1$, as $(n-F_{i-1})F_{i+1}=(j-F_i)F_i$. Because the Fibonacci numbers have no non-trivial common factors this is only satisfied for $n=F_{i-1}+mF_i$, $j=F_i+mF_{i+1}$, where m is an integer. The smallest m for which (23) holds is $m=1$, so $n=F_{i+1}$, $j=F_{i+2}$.

When the second term in (23) takes the value two, it can be rewritten, using $F_{i-3}F_{i+1} = F_{i-2}F_i + 2$, as $(n - F_{i-3})F_{i+1} = (j - F_{i-2})F_i$. For the same reason as above, this is only satisfied for $n = F_{i-3} + mF_i$, $j = F_{i-2} + mF_{i+1}$. Now as the smallest m for which (23) holds is $m = 2$, this gives greater values for n and j than in the first case.

When the second term $\geqslant 3$, j has to be greater than $3F_i$ but this is greater than F_{i+2}. So $n = F_{i+1}$ and $j = F_{i+2}$ is the first occurrence.

Result 2. ${T_i}^j{}_0 0$ and ${T_{i-1}}^j{}_0 0$ are on the same side of the cusp in $D(\theta)$ (2), for $j < F_{i+1}$.

Proof. First treat the *critical* case $K = 1$.

With $j < F_{i+1}$, it is impossible that points ${T_i}^j{}_0 0$ lie in the flat regions of figure 1(a) $(0 \leqslant \bar{\theta} \leqslant (a-1)/a)$, for if this happened, there would be a cycle of length j instead of F_{i+1} and another winding number would arise (Kadanoff 1983). So all points ${T_i}^j{}_0 0$ lie in the regions $(a-1)/a < \bar{\theta} < 1$.

In the *subcritical* K region $(0 \leqslant K < 1)$, I confine myself to a special value of K, $K = \kappa$ (6) to be determined later, such that

$$T_\infty{}^1{}_0 \tfrac{1}{2} = 1 \qquad a = 2. \tag{24}$$

The statement that has to be proved can now be written as: for all $j < F_{i+1}$, there is an n such that

$$n - \tfrac{1}{2} \leqslant {T_i}^j{}_0 0 \leqslant n + \tfrac{1}{2} \tag{25a}$$

and

$$n - \tfrac{1}{2} \leqslant {T_{i-1}}^j{}_0 0 \leqslant n + \tfrac{1}{2}. \tag{25b}$$

As intermediate steps I need

$$n + {T_i}^{-1}{}_0 0 \leqslant {T_i}^j{}_0 0 \leqslant n + 1 + {T_i}^{-1}{}_0 0 \tag{25c}$$

and

$$n + {T_{i-1}}^{-1}{}_0 0 \leqslant {T_{i-1}}^j{}_0 0 \leqslant n + 1 + {T_{i-1}}^{-1}{}_0 0. \tag{25d}$$

The fact that there is an n, for all $j < F_{i+1}$, such that (25c) and (25d) both hold, follows from the fact that ${T_i}^{j+1}{}_0 0$ and ${T_{i-1}}^{j+1}{}_0 0$ are in the same interval $[n, n+1]$ for $j < F_{i+2} - 1$ (see result 1).

I will now prove the equivalence between (25b) and (25d) for i even; the proof for i odd and for the equivalence of (25a) and (25c) is analogous. For i = even, $\Omega_\infty < \Omega_{i-1}$ (since $W_\infty < W_{i-1}$ and W is a monotonic function of Ω (Shenker 1982)), so ${T_{i-1}}^{-1}{}_0 < -\tfrac{1}{2}$. For the equivalence to hold there should not be any points ${T_{i-1}}^j{}_0 0$ in the regions not common to both (25b) and (25d); there should be no point ${T_{i-1}}^j{}_0 0$ such that

$$n + {T_{i-1}}^{-1}{}_0 0 < {T_{i-1}}^j{}_0 0 < n - \tfrac{1}{2}. \tag{26}$$

Applying T_{i-1} on all three terms and using (1) and (24) this is equivalent to

$$n < {T_{i-1}}^{j+1}{}_0 0 < n + T_{i-1\,0} - \tfrac{1}{2} = n + \Omega_{i-1} - \Omega_\infty. \tag{27}$$

So there is a forbidden interval for ${T_{i-1}}^{j+1}{}_0 0$. Now I will show that two points ${T_{i-1}}^j{}_0 0|_{\mathrm{mod}\ 1}$, that are nearest neighbours on the unit interval, will lie on different sides of the forbidden interval, so no point lies in it and the equivalence will be proven.

For $j = F_i - 1$, $T_{i-1}{}^{j+1}{}_0 0|_{\mathrm{mod}\,1}$ takes the value 0, so it is on the left-hand border of the forbidden interval. The points $T_{i-1}{}^{j}{}_0 0|_{\mathrm{mod}\,1}$ fall on the unit interval in exactly the same order as the points $jW_{i-1}|_{\mathrm{mod}\,1}$ (from (21), Kadanoff (1983)). On the unit interval the next point after $j = F_i - 1$ will be the one from $j = F_{i-1} - 1$, because

$$(F_{i-1} - 1) W_{i-1}|_{\mathrm{mod}\,1} - (F_i - 1) W_{i-1}|_{\mathrm{mod}\,1} = 1/F_i.$$

For $j = F_{i-1} - 1$, $T_{i-1}{}^{j+1}{}_0 0|_{\mathrm{mod}\,1} = T_{i-1}{}^{F_{i-1}}{}_0 0|_{\mathrm{mod}\,1} = T_{i-1}{}^{F_{i-1}}{}_0 0 - T_{i-2}{}^{F_{i-1}}{}_0 0$. The last equality follows from $\Omega_{i-1} > \Omega_{i-2}$, so the final term >0, and the final term is less than 1 from result 1. Now it has to be proved that

$$T_{i-1}{}^{F_{i-1}}{}_0 0 - T_{i-2}{}^{F_{i-1}}{}_0 0 > \Omega_{i-1} - \Omega_\infty \tag{28}$$

for the point to lie to the right of the forbidden interval.

From the mapping (1) it is clear that

$$T_{i-1}{}^{j}{}_0 0 - T_{i-2}{}^{j}{}_0 0 \geqslant \Omega_{i-1} - \Omega_{i-2} \tag{29}$$

for all $j > 0$, so also for $j = F_{i-1}$, and $\Omega_{i-1} - \Omega_{i-2} > \Omega_{i-1} - \Omega_\infty$. Hence (28) is satisfied.

So there always is an n for all j, such that (25*b*) and (25*d*) both hold. The same is true for (25*a*) and (25*c*). Furthermore, (25*c*) and (25*d*) are equivalent for $j < F_{i+1}$.

Result 3. The angle Φ_i, as defined in (12), is so small that $T_i{}^{j}{}_0\Phi_i$ will be in the same interval as $T_i{}^{j}{}_0 0$, and on the same side of the cusp in $D(\theta)$, as long as $j < F_{i-1}$.

Proof. Results 1 and 2 state that $T_i{}^{j}{}_0 0$ and $T_{i-1}{}^{j}{}_0 0$ are in the same interval and on the same side of the cusp for $j < F_{i+1}$. This holds also for $T_i{}^{F_i+j}{}_0 0$ and $T_{i-1}{}^{F_i+j}{}_0 0$ for $j < F_{i-1}$. $T_{i-1}{}^{F_i+j}{}_0 0$ can be written as $T_{i-1}{}^{j}{}_0 0 + F_{i-1}$. So $T_i{}^{F_i+j}{}_0 0 - F_{i-1}$ and $T_{i-1}{}^{j}{}_0 0$ are in the same interval and on the same side of the cusp for $j < F_{i-1}$, and so are $T_{i-1}{}^{j}{}_0 0$ and $T_i{}^{j}{}_0 0$. From (12) it follows that

$$T_i{}^{F_i+j}{}_0 0 - F_{i-1} = T_i{}^{j}{}_0 \Phi_i. \tag{30}$$

This completes the proof.

3. Analytical expressions for Ω_i and δ_i for the critical case ($K = 1$)

In this section the recursion relations for Ω_i (8) and the δ_i (5) are calculated at $K = 1$.

It is easy to calculate the first few Ω_i:

$i = 0$: $T_0{}^1{}_0 0 = \Omega_0 = 0$ whence $\Omega_0 = 0$ (31*a*)

$i = 1$: $T_1{}^1{}_0 0 = \Omega_1 = 1$ whence $\Omega_1 = 1$ (31*b*)

$i = 2$: $T_2{}^2{}_0 0 = T_2{}^1{}_0 \Omega_2 = (1 + a)\Omega_2 + 1 - a = 1$ whence $\Omega_2 = a/(a+1)$. (31*c*)

When i is even the subsequent Ω_i are found from the recursion relations (20) and (31). The only unknown quantity is $C(F_{i-1})$, which is the product of the slopes in figure 1 each time T has been applied. The map T has been applied F_{i-1} times with slope a each time, cf figure 1(*a*) and (1) and (2). Therefore

$$C(F_{i-1}) = a^{F_{i-1}}. \tag{32}$$

When i is odd equation (20) cannot be used. This is a result of the fact that Φ_i, as defined in (12), is larger than $(a-1)/a$, because Φ_i is greater than zero and it is impossible for these points to lie in the flat regions, as discussed in § 2. As a result, the $C(F_{i-1})$ in equation (14) should be multiplied by $\Phi_i-(a-1)/a$ instead of Φ_i. This problem can be avoided by reordering (11):

$$T_i^{F_{i+1}}{}_0 0 = T_i^{F_i}{}_0 T_i^{F_{i-1}}{}_0 0 = F_i. \tag{33}$$

The recursion relation, analogous to (20), but obtained with (33), is

$$A_i = A_{i-1} + C(F_i)A_{i-2} \tag{34a}$$

$$B_i = B_{i-1} + C(F_i)B_{i-2}. \tag{34b}$$

Since T has been applied F_i times and the slope is a each time

$$C(F_i) = a^{F_i} \tag{35}$$

is the analogue of (32).

In the critical case the Ω_i can therefore be expressed as

$$\begin{aligned}
\Omega_i = A_i/B_i \qquad & A_0 = 0\\
& A_1 = 1\\
& A_i = a^{F_{i-1}}A_{i-1} + A_{i-2} && i\text{: even}\\
& \qquad\quad A_{i-1} + a^{F_i}A_{i-2} && i\text{: odd}
\end{aligned} \tag{36a}$$

$$\begin{aligned}
& B_0 = 1\\
& B_1 = 1\\
& B_i = a^{F_{i-1}}B_{i-1} + B_{i-2} && i\text{: even}\\
& \qquad\quad B_{i-1} + a^{F_i}B_{i-2} && i\text{: odd.}
\end{aligned} \tag{36b}$$

Finally, having obtained these exact values for Ω_i it is easy to calculate δ_i (5):

$$\begin{aligned}
\delta_i &= -(a^{F_{i+2}}-1)/(a^{F_{i+2}}-a^{F_{i+1}}) && i\text{: even}\\
&\quad -(a^{F_{i+2}}-1)/(a^{F_i}-1) && i\text{: odd.}
\end{aligned} \tag{37}$$

Hence, for $i\to\infty$

$$\begin{aligned}
\lim_{i\to\infty}\delta_i &= -1 && i\text{: even}\\
\lim_{i\to\infty}\delta_i &= -\infty && i\text{: odd.}
\end{aligned} \tag{38}$$

4. Analytical expressions for Ω_i and δ_i in the subcritical case with $K=\kappa(<1)$

In this section I calculate the Ω_i and δ_i at $K=\kappa(6)$ and $a=2$. As has already been pointed out in (24), I study the case where

$$T_\infty{}^1{}_0\tfrac{1}{2} = 1 \qquad a = 2.$$

It appears later in this section that this condition is satisfied if and only if $K=\kappa$ where κ satisfies (6).

As in the critical case is it easy to calculate the first few Ω_i:

$$i=0\text{: } T_0{}^1{}_0 0 = \Omega_0 = 0 \qquad \text{whence } \Omega_0 = 0 \tag{39a}$$

$$i=1\text{: } T_1{}^1{}_0 0 = \Omega_1 = 1 \qquad \text{whence } \Omega_1 = 1 \tag{39b}$$

$$i=2\text{: } T_2{}^2{}_0 0 = (2+\kappa)\Omega_2 - \kappa = 1 \qquad \text{whence } \Omega_2 = (1+\kappa)/(2+\kappa). \tag{39c}$$

Again (20) can be used to calculate the Ω_i. The only problem is to calculate $C(F_{i-1})$.

When T_i is applied F_{i-1} times to the starting point, $\theta = 0$, the orbit has F_{i-3} points in the intervals $[n, n+\frac{1}{2}]$ and F_{i-2} points in the intervals $[n+\frac{1}{2}, n+1]$, $n \in N$. This is a result of the uniform distribution of the points in the pure rotation case, and the ordering properties following from (21). The real starting point for the $T_i^{F_{i-1}}$ lies in the second interval when $\theta < 0$ (i is even) and in the first interval when $\theta > 0$ (i is odd), cf (12). The last (F_{i-1}th) point, which has no influence on $C(F_{i-1})$, is in the first interval when i is even and in the second interval when i odd, as a result of the fact that jF_i/F_{i+1} for $j = F_{i-1}$, equals $F_{i-1}F_i/F_{i+1} = F_{i-2} + (-1)^i/F_{i+1}$. This lies in the interval $[n, n+\frac{1}{2}]$ if i is even, and in $[n+\frac{1}{2}, n+1]$ if i is odd. As a result

$$C(F_{i-1}) = (1-\kappa)^{F_{i-3}-1}(1+\kappa)^{F_{i-2}+1} \qquad i\text{: even}$$

$$(1-\kappa)^{F_{i-3}+1}(1+\kappa)^{F_{i-2}-1} \qquad i\text{: odd.} \tag{40}$$

In the subcritical case the Ω_i can therefore be expressed as

$$\Omega_i = A_i/B_i \qquad A_0 = 0$$

$$A_1 = 1$$

$$A_2 = 1+\kappa$$

$$A_i = A_{i-2} + (1-\kappa)^{F_{i-3}\pm 1}(1+\kappa)^{F_{i-2}\pm -1}A_{i-1} \tag{41a}$$

$$B_0 = 1$$

$$B_1 = 1$$

$$B_2 = 2+\kappa$$

$$B_i = B_{i-2} + (1-\kappa)^{F_{i-3}\pm 1}(1+\kappa)^{F_{i-2}\pm -1}B_{i-1} \tag{41b}$$

(i is odd: upper sign, i is even: lower sign).

One easily proves that the mapping has a winding number which is independent of the starting points (Kadanoff 1983). Therefore the distance between two θ points must remain finite. This distance is multiplied by $C(F_{i-1})$ after F_{i-1} mappings. Hence, I require

$$\lim_{i\to\infty} C(F_{i-1}) = \lim_{i\to\infty}(1-\kappa)^{F_{i-3}\pm 1}(1+\kappa)^{F_{i-2}\pm -1} = L < \infty. \tag{42}$$

This yields, taking the natural logarithm,

$$\lim_{i\to\infty}(F_{i-3}\pm 1)\ln(1-\kappa) = \lim_{i\to\infty}[\ln(L) - (F_{i-2}\pm -1)\ln(1+\kappa)]. \tag{43}$$

When dividing both sides in (43) by $F_{i-3}\pm 1$ and taking $i\to\infty$, the first term on the right-hand side will vanish. Note that

$$\lim_{i\to\infty}(F_{i-2}\pm -1)/(F_{i-3}\pm 1) = (1+\sqrt{5})/2. \tag{44}$$

Thus, I find the value of κ from

$$\ln(1-\kappa)/\ln(1+\kappa) = -(1+\sqrt{5})/2. \tag{45}$$

Finally, having obtained the exact results for Ω_i, it is easy to calculate δ_i (5):

$$\begin{aligned}\delta_i &= [\Omega_{i-1}(\kappa) - \Omega_i(\kappa)]/[\Omega_i(\kappa) - \Omega_{i+1}(\kappa)] \\ &= [(A_{i-1}B_i - A_iB_{i-1})B_{i+1}]/[(A_iB_{i+1} - A_{i+1}B_i)B_{i-1}] \\ &= -B_{i+1}/B_{i-1}\end{aligned} \tag{46}$$

using (41) or (20).

Using the κ value (45)

$$\lim_{i\to\infty}(1-\kappa)^{F_{i-3}\pm 1}(1+\kappa)^{F_{i-2}\pm -1} = [(1-\kappa)/(1+\kappa)]^{\pm 1}. \tag{47}$$

Hence, from (41*b*), for $i\to\infty$:

$$i\to\infty\colon\ B_i = B_{i-2} + (1-\kappa)^{\pm 1}(1+\kappa)^{\pm -1}B_{i-1}. \tag{48}$$

Thus

$$i\to\infty\colon\ B_i/B_{i-1} = (1-\kappa)^{\pm 1}(1+\kappa)^{\pm -1}(1+\sqrt{5})/2. \tag{49}$$

Finally,

$$\delta_\infty = \lim_{i\to\infty} -B_{i+1}/B_{i-1} = -(3+\sqrt{5})/2. \tag{50}$$

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